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EXACT THREE-DIMENSIONAL ELASTICITY SOLUTIONS FOR BENDING OF MODERATELY THICK INHOMOGENEOUS AND LAMINATED STRIPS UNDER NORMAL PRESSURE

T. G. ROGERS, P. WATSON and A. J. M. SPENCER
Department of Theoretical Mechanics, University of Nottingham, Nottingham NG7 2RD, U.K.

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Abstract—An exact three-dimensional solution is presented for the deformation and stress distribution in a semi-infinite strip clamped along its two edges, and subjected to uniform normal loading of the lateral surfaces. The strip is of constant moderate thickness and composed of anisotropic elastic material which is arbitrarily inhomogeneous in the through-thickness direction. The only material symmetry assumed is that of reflectional elastic symmetry in planes parallel to the mid-plane. The important special case of an anisotropic laminated plate is given by assuming piecewise-constant properties through the thickness.

The general method of solution is to reformulate the full three-dimensional elasticity equations in a way that reduces the problem to solving a system of partial differential equations in the two in-plane independent variables only, and then obtaining asymptotic solutions in terms of an aspect ratio of the thickness divided by a typical in-plane length. In general, successive terms are expressed in terms of the approximate "classical laminate" solution. In the present problem, this solution is very simple and the expansion terminates after no more than three terms. This gives a closed-form analytical solution that is valid for any aspect ratio.

1. INTRODUCTION

There are relatively few exact solutions to the full equations of the three-dimensional elastic theory as applied to the deformation of anisotropic plates and shells, and of these nearly all are in the context of homogeneous materials. Two recent solutions for general through-thickness inhomogeneity are given by Basi *et al.* (1991) and Rogers *et al.* (1992). Other solutions involve laminated materials, a special case of inhomogeneous materials for which the elastic moduli are piecewise-constant through the thickness of the plate or strip. Apart from some simple solutions involving homogeneous deformations and uniform shear stresses, the only exact solutions for laminates hitherto available appear to be those of Pagano (1969), (1970) on laminated strips and rectangular plates under sinusoidal normal pressures and surface displacements, and by Fan and Ye (1990) for a simply supported, uniformly loaded square plate.

The various approximate solutions which have been obtained for anisotropic laminates are based mainly on the "classical laminate theory" [see, for example, Calcote (1975), Jones (1975), Christensen (1979) and Whitney (1987)] or variants of that theory. This is the anisotropic equivalent of the conventional, but approximate, isotropic "thin-plate" theory and assumes the Kirchhoff–Love hypothesis of straight inextensible normals. It is found to predict displacements of thin anisotropic plates which agree well with experimental observations, and has been made the basis of many software packages. Unfortunately, it yields only average through-thickness values for the in-plane stresses and gives no information about the important interlaminar shear tractions (which are believed to be an indicator for possible delamination).

In this paper we apply a method due to Watson (1991) and Rogers *et al.* (1992) which provides accurate through-thickness solutions for materials with arbitrary through-thickness variation of the elastic moduli. The only restriction on the material response is that it has reflectional elastic symmetry in planes parallel to the lateral surfaces of the strip, i.e. it is monoclinic. It is *not* necessary, nor is it assumed, that the material inhomogeneity is symmetric about the mid-plane.

In the next section we show how the full system of equations describing three-dimensional elasticity may be reformulated in a way that reduces any three-dimensional plate problem to solving a system of partial differential equations in terms of the two in-plane independent variables only. The general method of solution, briefly described in section 3, consists of obtaining asymptotic expansions in terms of the aspect ratio ε of plate thickness divided by a typical length in the plane of the plate.

The equations relevant to a loaded strip are given in section 4. These are simpler than the general plate case, and have been termed cylindrical bending by Whitney (1987). In general, the solutions are not restricted to plane strain. For illustration, in section 5 we derive the full three-dimensional solution for a strip clamped along its two sides and loaded by a constant normal pressure applied to one face of the strip.

The method provides exact solutions whenever the problem is one for which the asymptotic expansion terminates after a finite number of terms, and these are then valid for *any* aspect ratio. This is the situation for the particular case presented in this paper. Furthermore, as with all solutions provided by the method, the leading term in the expansion is the classical laminate solution and successive terms are expressed in terms of this solution. For the present application, the classical laminate solution takes a particularly simple form, and the asymptotic expansion terminates after only three terms.

Solutions for *laminated* strips are obtained as special cases of the preceding analysis, and these are discussed in section 6. The method automatically ensures continuity of displacement and traction across the interlaminar surfaces. Some numerical results are presented in the final section.

2. REFORMULATION OF THE GOVERNING EQUATIONS

In our analysis, we refer all vector and tensor components to the rectangular cartesian coordinates x_1, x_2, x_3 with the mid-plane of the plate coinciding with $x_3 = 0$. The displacement $\mathbf{u}$ at any point $\mathbf{x}$ has components u_i and the stress tensor $\boldsymbol{\sigma}$ has components σ_{ij}. Here, and in the remainder of the paper, Latin subscripts take the values 1, 2 or 3 unless otherwise stated, whilst Greek subscripts take values 1 or 2; we also adopt the usual summation convention that a repeated suffix implies summation of its entire range.

The anisotropic elastic response is that of monoclinic linear elasticity, for which the constitutive equations are

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,3}+u_{3,2} \\ u_{1,3}+u_{3,1} \\ u_{1,2}+u_{2,1} \end{bmatrix}. \tag{1}$$

Here commas denote partial differentiation with respect to the relevant position coordinates x_1, x_2, x_3 (so $u_{1,3} \equiv \partial u_1/\partial x_3$, etc.). The number of independent elastic moduli c_{ij} is reduced from thirteen to nine if the behaviour is orthotropic with respect to the x_1, x_2, x_3 axes, and to five if the material is transversely isotropic with respect to the x_1-axis (for example).

If the material is either orthotropic or transversely isotropic, with a symmetry axis which makes an angle φ with the x_1-axis (as would be the case for most commercially produced laminates), then its stress–strain representation still takes the form given in eqn (1); however, the thirteen moduli would then be related to φ and the nine or five independent moduli of the respective anisotropy.

The inhomogeneity is with respect to x_3 only, so

$$c_{ij} = c_{ij}(x_3), \quad i, j = 1, 2, \dots 6.$$

For the particular case of a laminate, the c_{ij} are constant within each layer and are accordingly piecewise-constant functions of x_3.

The only other governing equations are those of equilibrium, which for negligible body forces take the form

$$\begin{aligned} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} &= 0 \\ \sigma_{12,1} + \sigma_{22,2} + \sigma_{23,3} &= 0 \\ \sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} &= 0. \end{aligned} \tag{2}$$

For convenience, we introduce the dimensionless variables

$$U_i = u_i/l, \quad \tau_{ij} = \sigma_{ij}/c^*, \quad X_\alpha = x_\alpha/l, \quad X_3 = x_3/h \tag{3}$$

where l is a typical in-plane length, c^* is a typical stiffness modulus and $2h$ is the thickness of the plate. Hence a dimensionless thickness parameter

$$\varepsilon = h/l \tag{4}$$

is introduced. In practice this is usually small compared with unity, and is a suitable parameter for the asymptotic expansions used in the analysis.

In order that none of the moduli c_{ij} $(i,j = 1,2,\dots 6)$ need be differentiated with respect to x_3 (which would cause difficulties at the interfaces within a laminate), the nine governing eqns (1) and (2) are rearranged into the form

$$\frac{\partial}{\partial X_3}\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} = \varepsilon \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} \tag{5}$$

and

$$\mathbf{H} = \mathbf{CF}. \tag{6}$$

Here **F**, **G** and **H** are defined as

$$\mathbf{F} = \begin{bmatrix} U_1 \\ U_2 \\ \tau_{33} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \tau_{13} \\ \tau_{23} \\ U_3 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{bmatrix} \tag{7}$$

and **A**, **B** and **C** are matrix operators [see Rogers *et al.* (1992)] which include differential operators with respect to X_1 and X_2 but *not* X_3; they also involve the material moduli c_{ij} which provide the only dependence on X_3. The forms relevant to strip problems are given explicitly in section 4.

Formally, eqn (5) has the solution [see, for example, Gantmacher (1960)]

$$\begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{F}^* \\ \mathbf{G}^* \end{bmatrix}, \tag{8}$$

where **P** is a matrix operator defined below, and where the reference functions **F*** and **G*** are functions of X_1 and X_2 only, being defined as the values of **F** and **G** at some reference plane $X_3 = X_3^*$. These six scalar functions $F_i^*, G_i^* (i = 1, 2, 3)$ are then the unknowns that need to be determined such that the boundary conditions specified on the two lateral surfaces $X_3 = \pm 1$ are satisfied. They are also required to be such that the edge conditions on the plate are satisfied in the usual average through-thickness sense assumed in all plate theories.

Most plate theories, including the classical laminate theory, implicitly designate the reference plane $X_3 = X_3^*$ to be the mid-plane of the plate, i.e. $X_3^* = 0$. However, it is clear from eqn (8) that if the reference plane is chosen to be one or other of the plate surfaces $X_3 = \pm 1$, then at least three of $F_i^*$and G_i^*will be known *a priori*, so reducing the problem to determining the remaining (at most three) unknown functions. Thus for problems in which the boundary tractions are specified, for example, the values of τ_{13}, τ_{23} and τ_{33} are known at $X_3 = X_3^* = -1$ (say). The unknown functions are then only U_1, U_2 and U_3 on the lower surface, and they have to be chosen in such a way that the traction conditions on the upper surface $X_3 = +1$ are satisfied.

The transfer matrix $\mathbf{P}$ of eqn (8) is given by

$$\mathbf{P} = \sum_{n=0}^{\infty} \varepsilon^{2n} \begin{bmatrix} \mathbf{a}^{(2n)} & \varepsilon \mathbf{a}^{(2n+1)} \\ \varepsilon \mathbf{b}^{(2n+1)} & \mathbf{b}^{(2n)} \end{bmatrix}, \tag{9}$$

where

$$\mathbf{a}^{(k-1)}(X_3) = \int_{X_3^*}^{X_3} \mathbf{A}(\xi)\mathbf{b}^{(k)}(\xi)\,\mathrm{d}\xi, \tag{10}$$

$$\mathbf{b}^{(k+1)}(X_3) = \int_{X_3^*}^{X_3} \mathbf{B}(\xi)\mathbf{a}^{(k)}(\xi)\,\mathrm{d}\xi, \tag{11}$$

for $k = 0, 1, \ldots$, with $\mathbf{a}^{(0)} = \mathbf{b}^{(0)} = \mathbf{I}$, the identity matrix. The variation with X_3 shown in eqns (10) and (11) appears only in the material moduli; just as for $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, the $\mathbf{a}^{(k)}$ and $\mathbf{b}^{(k)}$ are matrix operators which include differential operators with respect to X_1 and X_2 but not X_3.

3. EXPANSION SOLUTION

Choosing $X_3^* = -1$ and then imposing the remaining boundary conditions at $X_3 = +1$ onto the solution given by eqn (8) produces three scalar differential equations of infinite order for the three unknown reference functions, though only X_1 and X_2 are now involved. By expanding these reference functions as power series in ε, it is shown in Watson (1991) that the problem finally reduces to finding a sequence of particular integrals for the classical plate equations. Thus we substitute

$$\mathbf{F}^* = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{f}^{(n)}(X_1, X_2), \quad \mathbf{G}^* = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{g}^{(n)}(X_1, X_2), \tag{12}$$

into eqn (8) to give

$$\begin{aligned} \mathbf{F} = \mathbf{f}^{(0)} &+ \varepsilon[\mathbf{f}^{(1)} + \mathbf{a}^{(1)}\mathbf{g}^{(0)}] + \varepsilon^2[\mathbf{f}^{(2)} + \mathbf{a}^{(1)}\mathbf{g}^{(1)} + \mathbf{a}^{(2)}\mathbf{f}^{(0)}] \\ &+ \varepsilon^3[\mathbf{f}^{(3)} + \mathbf{a}^{(1)}\mathbf{g}^{(2)} + \mathbf{a}^{(2)}\mathbf{f}^{(1)} + \mathbf{a}^{(3)}\mathbf{g}^{(0)}] + \cdots, \\ \mathbf{G} = \mathbf{g}^{(0)} &+ \varepsilon[\mathbf{g}^{(1)} + \mathbf{b}^{(1)}\mathbf{f}^{(0)}] + \varepsilon^2[\mathbf{g}^{(2)} + \mathbf{b}^{(1)}\mathbf{f}^{(1)} + \mathbf{b}^{(2)}\mathbf{g}^{(0)}] \\ &+ \varepsilon^3[\mathbf{g}^{(3)} + \mathbf{b}^{(1)}\mathbf{f}^{(2)} + \mathbf{b}^{(2)}\mathbf{g}^{(1)} + \mathbf{b}^{(3)}\mathbf{f}^{(0)}] + \cdots. \end{aligned} \tag{13}$$

Recalling that the $\mathbf{a}^{(k)}$ and $\mathbf{b}^{(k)}$ depend on X_3 through eqns (10) and (11), we then substitute the appropriate boundary values at $X_3 = +1$ into the relevant components of eqn (13) and equate corresponding coefficients of powers of ε.

For stress boundary value problems this technique shows that for general loading conditions (in particular, when the tractions on both surfaces are not equal) the shear stresses are $0(\varepsilon^2)$ and the normal stress is $0(\varepsilon^3)$. Then one of the equations of order ε is initially satisfied, and the remaining two are

$$b_{11}^{(1)}U_1^{(0)}+b_{12}^{(1)}U_2^{(0)}=0\,,\quad b_{21}^{(1)}U_1^{(0)}+b_{22}^{(1)}U_2^{(0)}=0. \tag{14}$$

One of the equations of order ε^2 gives a further relation between $U_1^{(0)}$ and $U_2^{(0)}$, namely

$$a_{31}^{(2)}U_1^{(0)}+a_{32}^{(2)}U_2^{(0)}=0. \tag{15}$$

In each of these equations, the coefficients have been evaluated at $X_3 = 1$. In fact, the coefficients in eqns (14) and (15) can be shown to be differential operators of orders two and three respectively, and hence $U_\alpha^{(0)}$ can only be linear functions of X_1 and X_2.

The remaining two equations of order ε^2, and one of order ε^3, provide the system:

$$\begin{aligned}
b_{11}^{(1)}U_1^{(1)}+b_{12}^{(1)}U_2^{(1)}+b_{13}^{(2)}U_3^{(0)} &= s_1^+-s_1^-,\\
b_{21}^{(1)}U_1^{(1)}+b_{22}^{(1)}U_2^{(1)}+b_{23}^{(2)}U_3^{(0)} &= s_2^+-s_2^-,\\
a_{31}^{(2)}U_1^{(1)}+a_{32}^{(2)}U_2^{(1)}+a_{33}^{(3)}U_3^{(0)} &= -p^++p^--a_{31}^{(1)}s_1^--a_{32}^{(1)}s_2^-,
\end{aligned} \tag{16}$$

where the surface tractions have been scaled according to

$$\tau_{\alpha 3}(X_1,X_2,\pm 1)=\varepsilon^2 s_\alpha^\pm\,,\quad \tau_{33}(X_1,X_2,\pm 1)=-\varepsilon^3 \mathrm{p}^\pm. \tag{17}$$

Continuing the process to higher orders of ε gives the following systems of equations

$$\begin{aligned}
b_{11}^{(1)}U_1^{(2)}+b_{12}^{(1)}U_2^{(2)}+b_{13}^{(2)}U_3^{(1)} &= 0,\\
b_{21}^{(1)}U_1^{(2)}+b_{22}^{(1)}U_2^{(2)}+b_{23}^{(2)}U_3^{(1)} &= 0,\\
a_{31}^{(2)}U_1^{(2)}+a_{32}^{(2)}U_2^{(2)}+a_{33}^{(3)}U_3^{(1)} &= 0,
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
b_{11}^{(1)}U_1^{(3)}+b_{12}^{(1)}U_2^{(3)}+b_{13}^{(2)}U_3^{(2)} &= -b_{11}^{(3)}U_1^{(1)}-b_{12}^{(3)}U_2^{(1)}-b_{13}^{(4)}U_3^{(0)}-b_{11}^{(2)}s_1^--b_{12}^{(2)}s_2^-+b_{13}^{(1)}p^-,\\
b_{21}^{(1)}U_1^{(3)}+b_{22}^{(1)}U_2^{(3)}+b_{23}^{(2)}U_3^{(2)} &= -b_{21}^{(3)}U_1^{(1)}-b_{22}^{(3)}U_2^{(1)}-b_{23}^{(4)}U_3^{(0)}-b_{21}^{(2)}s_1^--b_{22}^{(2)}s_2^-+b_{23}^{(1)}p^-,\\
a_{31}^{(2)}U_1^{(3)}+a_{32}^{(2)}U_2^{(3)}+a_{33}^{(3)}U_3^{(2)} &= -a_{31}^{(4)}U_1^{(1)}-a_{32}^{(4)}U_2^{(1)}-a_{33}^{(5)}U_3^{(0)}-a_{31}^{(3)}s_1^--a_{32}^{(3)}s_2^-+a_{33}^{(2)}p^-.
\end{aligned} \tag{19}$$

We note that the left hand sides of all of the above systems contain the same coefficients. Furthermore, a more detailed examination of these coefficients would reveal that they are in fact precisely the same as the coefficients of the coupled classical equations [see, for example, Christensen (1979)], provided those equations are rewritten in terms of the displacements of the lower surface (rather than those of the mid-plane). Hence the leading order terms in our solution are provided by the classical theory. Moreover, all the terms on the right hand sides of all the systems such as those in eqns (16) and (19) above are expressed as derivatives of the lower order solutions. Thus the remaining terms in the solution itself must also be expressible in terms of derivatives of the lowest order solution, which is the classical laminate solution. Accordingly, if the two-dimensional classical laminate solution has been obtained for any particular problem, then the above procedure may be used to generate the accurate three-dimensional solution for that same problem. Furthermore, this is always the case whether the two-dimensional solution is an analytical expression, for example, or has been computed by a numerical procedure (finite elements, finite differences, etc.).

4. STRIP PROBLEMS

A strip $|x_1| \leqslant l$ is a suitable model for a rectangular plate $|x_1| \leqslant l$, $|x_2| \leqslant a$ for which the length $2a$ is much greater than its width $2l$. The solution for any strip problem is thus effectively independent of x_2. Then the matrix operators defined in the previous two sections take the simpler forms

$$\mathbf{A} = \begin{bmatrix} Q_{44} & -Q_{45} & -\partial \\ -Q_{45} & Q_{55} & 0 \\ -\partial & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = -\begin{bmatrix} Q_{11}\partial^2 & Q_{16}\partial^2 & Q_{13}\partial \\ Q_{16}\partial^2 & Q_{66}\partial^2 & Q_{36}\partial \\ Q_{13}\partial & Q_{36}\partial & -Q_{33} \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} Q_{11}\partial & Q_{16}\partial & Q_{13} \\ Q_{12}\partial & Q_{26}\partial & Q_{23} \\ Q_{16}\partial & Q_{66}\partial & Q_{36} \end{bmatrix}, \tag{20}$$

where $\partial \equiv \partial/\partial X_1$ and the Q_{ij} are dimensionless material moduli defined by

$$Q_{ij} = \begin{cases} (c_{ij} - c_{i3}c_{j3}/c_{33})/c^*, & i, j = 1, 2, 6 \\ c^* c_{ij}/(c_{44}c_{55} - c_{45}^2), & i, j = 4, 5 \end{cases} \tag{21}$$

$$Q_{i3} = c_{i3}/c_{33}, \quad i = 1, 2, 6; \quad Q_{33} = c^*/c_{33}.$$

From eqns (10) and (11) we obtain the matrix operators

$$a_{3i}^{(1)}(X_3) = \int_{-1}^{X_3} A_{3i}(\xi)\,\mathrm{d}\xi, \quad b_{\alpha\beta}^{(1)}(X_3) = \int_{-1}^{X_3} B_{\alpha\beta}(\xi)\,\mathrm{d}\xi$$

so that, with $X_3 = +1$, the relevant coefficients in eqns (16), (18) and (19) are

$$a_{31}^{(1)} = -2\partial, \quad a_{32}^{(1)} = a_{33}^{(1)} = 0,$$

$$b_{11}^{(1)} = -\bar{Q}_{11}\partial^2, \quad b_{12}^{(1)} = b_{21}^{(1)} = -\bar{Q}_{16}\partial^2, \quad b_{22}^{(1)} = -\bar{Q}_{66}\partial^2$$

where

$$\bar{Q}_{ij} = \int_{-1}^{1} Q_{ij}(X_3)\,\mathrm{d}X_3. \tag{22}$$

Similarly, further application of eqns (10) and (11) gives

$$a_{31}^{(2)} = (\bar{Q}_{11} - \tilde{Q}_{11})\partial^3, \quad a_{32}^{(2)} = (\bar{Q}_{16} - \tilde{Q}_{16})\partial^3,$$

$$b_{13}^{(2)} = (\bar{Q}_{11} + \tilde{Q}_{11})\partial^3, \quad b_{23}^{(2)} = (\bar{Q}_{16} + \tilde{Q}_{16})\partial^3,$$

$$a_{33}^{(3)} = (\hat{Q}_{11} - \bar{Q}_{11})\partial^4$$

where

$$\tilde{Q}_{ij} = \int_{-1}^{1} X_3 Q_{ij}(X_3)\,\mathrm{d}X_3, \quad \hat{Q}_{ij} = \int_{-1}^{1} X_3^2 Q_{ij}(X_3)\,\mathrm{d}X_3. \tag{23}$$

The eqns (16) become, after a little manipulation,

$$\begin{aligned}
\bar{Q}_{11}\partial^2 U_1^{(1)} + \bar{Q}_{16}\partial^2 U_2^{(1)} - (\bar{Q}_{11} + \tilde{Q}_{11})\partial^3 U_3^{(0)} &= l_1^{(0)}, \\
\bar{Q}_{16}\partial^2 U_1^{(1)} + \bar{Q}_{66}\partial^2 U_2^{(1)} - (\bar{Q}_{16} + \tilde{Q}_{16})\partial^3 U_3^{(0)} &= l_2^{(0)}, \\
\tilde{Q}_{11}\partial^3 U_1^{(1)} + \tilde{Q}_{16}\partial^3 U_2^{(1)} - (\tilde{Q}_{11} + \hat{Q}_{11})\partial^4 U_3^{(0)} &= l_3^{(0)},
\end{aligned} \tag{24}$$

where $l_1^{(0)}, l_2^{(0)}$ and $l_3^{(0)}$ are defined in terms of the surface conditions, and are given by

$$\begin{aligned}
l_1^{(0)} &= -s_1^+ + s_1^-, \\
l_2^{(0)} &= -s_2^+ + s_2^-, \\
l_3^{(0)} &= p^+ - p^- - \partial s_1^+ - \partial s_1^-.
\end{aligned} \tag{25}$$

These are the same as the usual classical laminate equations for a strip, generalised to include non-zero shear conditions on both surfaces. Elimination of $U_1^{(1)}$ and $U_2^{(1)}$ yields

$$E\partial^4 U_3^{(0)} = -l_3^{(0)} + L_1 \partial l_1^{(0)} + L_2 \partial l_2^{(0)} \tag{26}$$

where the equivalent extensional modulus E is defined by

$$E = \hat{Q}_{11} - (\bar{Q}_{11}\tilde{Q}_{16}^2 - 2\bar{Q}_{16}\tilde{Q}_{11}\tilde{Q}_{16} + \bar{Q}_{66}\tilde{Q}_{11}^2)/\Delta \tag{27}$$

with

$$\Delta = \bar{Q}_{11}\bar{Q}_{66} - \bar{Q}_{16}^2, \tag{28}$$

and

$$\begin{aligned}
L_1 &= (\tilde{Q}_{11}\bar{Q}_{66} - \tilde{Q}_{16}\bar{Q}_{16})/\Delta, \\
L_2 &= -(\tilde{Q}_{11}\bar{Q}_{16} - \tilde{Q}_{16}\bar{Q}_{11})/\Delta.
\end{aligned} \tag{29}$$

Equation (26), taken with eqns (25) and (27)–(29), may be recognized as the standard equation governing the normal displacement of the strip, namely

$$E\partial^4 U_3^{(0)} = -p^+ + p^- - L_1[\partial s_1^+ - \partial s_1^-] - L_2[\partial s_2^+ - \partial s_2^-] + \partial s_1^+ + \partial s_1^-, \tag{30}$$

and E is the *equivalent extensional modulus* of the inhomogeneous or laminated strip.

We note that when the inhomogeneity is symmetric, with $c_{ij}(X_3) = c_{ij}(-X_3)$, then the $\tilde{Q}_{ij}$ are all zero and

$$E = \hat{Q}_{11}, \quad L_1 = L_2 = 0.$$

For the even more special case of a homogeneous, though still monoclinic, strip the through-thickness equivalent moduli simplify to

$$\bar{Q}_{ij} = 2Q_{ij}, \quad \tilde{Q}_{ij} = 0, \quad \hat{Q}_{ij} = \tfrac{2}{3}Q_{ij}$$

so that

$$E = \tfrac{2}{3}Q_{11}.$$

The standard strip equation (30) is a special case of classical laminate theory and is easily solved for $U_3^{(0)}(X_3)$ by simple integration of the right hand side, which is given in terms of the tractions imposed on the upper and lower surfaces of the strip. Applying the appropriate edge conditions at $X_1 = \pm 1$ then gives the relevant expression for $U_3^{(0)}$. Whenever an analytical solution of eqn (30) is not possible (for example, if the surface conditions are specified in the form of numerical data or complicated functions) then a numerical solution can be determined by straightforward quadratures.

From eqns (14) and (15) it is clear that $U_1^{(0)}$ and $U_2^{(0)}$ are linear functions of X_1 and, for boundary-value problems with fixed edges, they may conveniently be set to zero. The leading terms in U_1^* and U_2^* are then $\varepsilon U_1^{(1)}$ and $\varepsilon U_2^{(1)}$ which are obtained by substituting for $U_3^{(0)}$ into eqns (24) and (25) and then solving the resulting pair of simultaneous differential equations.

Equations (18) are satisfied by

$$U_3^{(1)} = 0, \quad U_\alpha^{(2)} = 0 \quad (\alpha = 1, 2),$$

so the next terms in U_3^* and U_α^* are $\varepsilon^2 U_3^{(2)}$ and $\varepsilon^3 U_\alpha^{(3)}$ respectively, which are given by solving the appropriate form of eqns (19). As already noted, these equations have the same structure as eqns (24) governing $U_3^{(0)}$ and $U_\alpha^{(1)}$. However, the latter are now replaced by $U_3^{(2)}$ and $U_\alpha^{(3)}$, and the right hand sides $l_1^{(0)}, l_2^{(0)}$ and $l_3^{(0)}$ are replaced by the much more complex $l_1^{(2)}, l_2^{(2)}$ and $l_3^{(2)}$, given by

$$\begin{aligned}
l_1^{(2)} &= \bar{Q}_{13}\partial^2 p^- + \bar{R}_{11}\partial^3 s_1^- + \bar{R}_{12}\partial^3 s_2^- - \bar{S}_{11}\partial^5 U_1^{(1)} - \bar{S}_{12}\partial^5 U_2^{(1)} + (\bar{S}_{11} + \check{S}_{11})\partial^6 U_3^{(0)} \\
l_2^{(2)} &= \bar{Q}_{36}\partial^2 p^- + \bar{R}_{21}\partial^3 s_1^- + \bar{R}_{22}\partial^3 s_2^- - \bar{S}_{21}\partial^5 U_1^{(1)} - \bar{S}_{22}\partial^5 U_2^{(1)} + (\bar{S}_{21} + \check{S}_{21})\partial^6 U_3^{(0)} \\
l_3^{(2)} &= \tilde{Q}_{13}\partial^2 p^- + \bar{R}_{11}\partial^3 s_1^- + \bar{R}_{12}\partial^3 s_2^- - \tilde{S}_{11}\partial^5 U_1^{(1)} - \tilde{S}_{12}\partial^5 U_2^{(1)} + (\tilde{S}_{11} + \check{S}_{21})\partial^6 U_3^{(0)}.
\end{aligned} \tag{31}$$

Here the various coefficients are all differently weighted integrals over the thickness of the strip, $-1 \leqslant X_3 \leqslant 1$, of various combinations of the inhomogeneous moduli Q_{ij}; these are defined and evaluated in the Appendix. The previous analysis shows that $U_3^{(2)}$ satisfies

$$E\partial^4 U_3^{(2)} = -l_3^{(2)} + L_1 \partial l_1^{(2)} + L_2 \partial l_2^{(2)}, \tag{32}$$

subject to homogeneous boundary conditions at $X_3 = \pm 1$.

Although the right hand side of eqn (32) is in general much more complicated than that in eqn (24), nevertheless it is expressed in terms of the leading order solution $U_3^{(0)}, \varepsilon U_1^{(1)}$ and $\varepsilon U_2^{(1)}$ and is therefore a known function of X_3. Hence $U_3^{(2)}$ is straightforward to determine. Then the equations may be solved for $U_1^{(3)}$ and $U_2^{(3)}$, and the procedure continued to any required degree of accuracy. In some cases, as with the particular problem considered in the following section, the procedure terminates and hence provides an *exact* solution.

5. CLAMPED STRIP UNDER UNIFORM PRESSURE

As an illustrative example, we consider the case (refer to Fig. 1) of a strip clamped along its edges $X_1 = \pm 1$ and subjected to a uniform pressure equivalent to $p^+ = p$ so that

$$s_1^+ = s_1^- = s_2^+ = s_2^- = p^- = 0, \quad p^+ = p. \tag{33}$$

For convenience, we interpret the "clamped edges" specification as applying along the edges of the reference surface $X_3 = -1$; hence

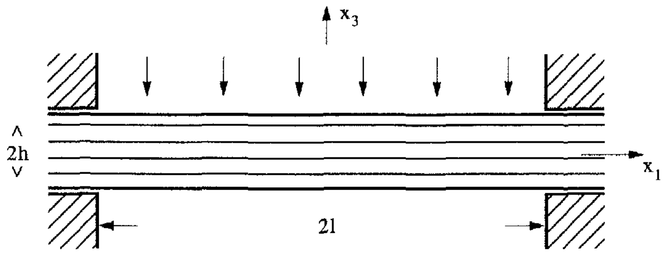

Fig. 1. Clamped laminated (or more generally inhomogeneous) strip subjected to constant normal pressure.

$$U_1^* = U_2^* = U_3^* = 0, \quad \partial U_3^*/\partial X_1 = 0 \quad \text{on} \quad X_1 = \pm 1. \tag{34}$$

If more detailed point-wise conditions are specified across the edges, then the solution needs to incorporate boundary layer solutions in the edge regions.

Conditions (33) and (34), when inserted into eqns (26) and (12), yield the familiar fourth order differential equation for $U_3^{(0)}$:

$$E\partial^4 U_3^{(0)} = -p, \tag{35}$$

subject to

$$U_3^{(0)} = \partial U_3^{(0)}/\partial X_1 = 0 \quad \text{on} \quad X_1 = \pm 1. \tag{36}$$

The elementary classical solution is

$$U_3^{(0)} = -k(1-X_1^2)^2 \tag{37}$$

with

$$k = \tfrac{1}{24}p/E. \tag{38}$$

The leading order terms for U_1 and U_2 are then determined from eqn (24) as

$$U_\alpha^{(1)} = k_\alpha X_1(1-X_1^2), \quad \alpha = 1,2 \tag{39}$$

with

$$\begin{aligned} k_1 &= 4k(1+L_1) \\ k_2 &= 4kL_2. \end{aligned} \tag{40}$$

The zero right hand sides of eqns (18), taken together with conditions (34), immediately give

$$U_3^{(1)} = 0, \quad U_1^{(2)} = U_2^{(2)} = 0. \tag{41}$$

Also, inspection of eqns (31) shows that in the present example the right hand sides of eqns (19) are zero:

$$l_1^{(2)} = l_2^{(2)} = l_3^{(2)} = 0.$$

Hence

$$U_3^{(2)} = 0, \quad U_1^{(3)} = U_2^{(3)} = 0,$$

and, since the orders of the differential operators on the right hand sides continue to increase with each successive power of ε, we have

$$U_3^{(n)} = 0, \quad U_1^{(n+1)} = U_2^{(n+1)} = 0, \quad n \geqslant 1.$$

Thus the reference-value evaluation is complete and the full through-thickness solution is obtained by substituting these values into eqn (13) and, for the in-plane stresses, into eqns (6) and (20). In particular, the exact analytical solution for the displacement is obtained explicitly as

$$U_1 = \varepsilon X_1(1-X_1^2)[k_1-4k(X_3+1)]+\varepsilon^3 X_1 \int_{-1}^{X_3}\int_{-1}^{\xi} N_1(\xi,\eta)\,d\eta\,d\xi,$$

$$U_2 = \varepsilon k_2 X_1(1-X_1^2)+\varepsilon^3 X_1 \int_{-1}^{X_3}\int_{-1}^{\xi} N_2(\xi,\eta)\,d\eta\,d\xi,$$

$$U_3 = -k(1-X_1^2)^2-\tfrac{1}{6}\varepsilon^2(1-3X_1^2)\int_{-1}^{X_3} N_3(\xi)\,d\xi-\varepsilon^4\int_{-1}^{X_3}\int_{-1}^{\xi}\int_{-1}^{\eta} N(\xi,\eta,\zeta)\,d\zeta\,d\eta\,d\xi, \tag{42}$$

and the out-of-plane stress components are given by

$$\tau_{13} = \varepsilon^2 X_1 \int_{-1}^{X_3} N_4(\xi)\,d\xi,$$

$$\tau_{23} = \varepsilon^2 X_1 \int_{-1}^{X_3} N_5(\xi)\,d\xi,$$

$$\tau_{33} = -\varepsilon^3 \int_{-1}^{X_3} (X_3-\xi)N_4(\xi)\,d\xi. \tag{43}$$

Here

$$N_1(\xi,\eta) = Q_{44}(\xi)N_4(\eta)-Q_{45}(\xi)N_5(\eta)-N_3(\eta),$$
$$N_2(\xi,\eta) = -Q_{45}(\xi)N_4(\eta)+Q_{55}(\xi)N_5(\eta),$$
$$N_3(\xi) = \{L_1Q_{13}(\xi)+L_2Q_{36}(\xi)-\xi Q_{13}(\xi)\}\,p/E,$$
$$N_4(\xi) = \{L_1Q_{11}(\xi)+L_2Q_{16}(\xi)-\xi Q_{11}(\xi)\}\,p/E,$$
$$N_5(\xi) = \{L_1Q_{16}(\xi)+L_2Q_{66}(\xi)-\xi Q_{16}(\xi)\}\,p/E, \tag{44}$$

and

$$N(\xi,\eta,\zeta) = Q_{13}(\xi)N_1(\eta,\zeta)+Q_{36}(\xi)N_2(\eta,\zeta)+Q_{33}(\xi)N_4(\zeta),$$

where L_1 and L_2 are the laminate constants given by eqn (29). The out-of-plane stresses in eqn (43) may be obtained alternatively by direct substitution of the displacement solution in eqn (42) into the non-dimensionalised form of the constitutive equations (1).

The in-plane stress components are obtained from eqns (42), (6) and (20) as

$$\tau_{11} = \varepsilon(1-3X_1^2)[k_2Q_{16}+\{k_1-4k(X_3+1)\}Q_{11}]+Q_{13}\tau_{33} + \varepsilon^3\int_{-1}^{X_3}\int_{-1}^{\xi}\{Q_{11}(X_3)N_1(\xi,\eta)+Q_{16}(X_3)N_2(\xi,\eta)\}\,d\eta\,d\xi,$$

$$\tau_{22} = \varepsilon(1-3X_1^2)[k_2Q_{26}+\{k_1-4k(X_3+1)\}Q_{12}]+Q_{23}\tau_{33} + \varepsilon^3\int_{-1}^{X_3}\int_{-1}^{\xi}\{Q_{12}(X_3)N_1(\xi,\eta)+Q_{26}(X_3)N_2(\xi,\eta)\}\,d\eta\,d\xi,$$

$$\tau_{12} = \varepsilon(1-3X_1^2)[k_2Q_{66}+\{k_1-4k(X_3+1)\}Q_{16}]+Q_{36}\tau_{33} + \varepsilon^3\int_{-1}^{X_3}\int_{-1}^{\xi}\{Q_{16}(X_3)N_1(\xi,\eta)+Q_{66}(X_3)N_2(\xi,\eta)\}\,d\eta\,d\xi, \quad (45)$$

where τ_{33} is given in eqn (43) and is of order ε^3. This completes the analytical, exact solution of the present problem.

Although the method of solution is applicable for any distribution of tractions on the lateral surfaces, the simplicity of the classical plate solution in eqns (37)–(40) in the present case results in an exact analytical solution which has a very simple dependence on X_1. In particular, the out-of-plane shear stress components σ_{13} and σ_{23}—about which the classical plate theory gives no information—are linear in X_1; hence they, and the resultant shear traction $\sqrt{(\sigma_{13}^2+\sigma_{23}^2)}$, have their maximum magnitudes along the edges $X_1 = \pm 1$ for all values of the material moduli. The analysis also shows that the non-dimensionalised normal stress component τ_{33} is independent of X_1 but varies from zero on the lower surface ($X_3 = -1$) to $-p$ on the upper surface ($X_3 = 1$) in a non-trivial way; again the classical theory gives no information for this stress component either.

Clearly, for arbitrary variation of inhomogeneity through the thickness, the dependence of the solution on X_3 is expressed through integrals. However, these integrals are all straightforward quadratures and are all independent of X_1. For laminated strips, as discussed in the next section, the integrals may be evaluated analytically as relatively simple continuous, piecewise polynomial functions of X_3.

6. LAMINATED STRIPS

Lamination is an important special case of inhomogeneity, with the material moduli c_{ij} then being discontinuous, but piecewise constant, functions of the through-thickness variable.

We consider a laminated strip consisting of N layers, perfectly bonded at their interfaces. Each layer is still monoclinic but is now homogeneous. For a typical rth layer, the thickness and material moduli are denoted by h_r and $c_{ij}^{(r)}$ respectively, and similarly any other quantity relating to the rth layer will also be identified by the index r. Hence

$$\sum_{r=1}^{N} h_r = 2h \quad (46)$$

or, in terms of the non-dimensional thicknesses $H_r(=h_r/h)$,

$$\sum_{r=1}^{N} H_r = 2. \quad (47)$$

Values at the various interfaces have particular importance in practice. We denote the interface between the $(r-1)$th layer and the adjoining rth layer by $X_3 = Z_r$. Then the rth layer is defined by $Z_r \leqslant X_3 \leqslant Z_{r+1}$, $r \geqslant 1$, and the lateral surfaces $X_3 = \pm 1$ are denoted by $Z_1 = -1$ and $Z_{N+1} = 1$. Thus

$$Q_{ij}(X_3) = Q_{ij}^{(r)}, \quad Z_r < X_3 < Z_{r-1}. \tag{48}$$

From eqn (44), it is immediately obvious that $N_3(\xi)$, $N_4(\xi)$ and $N_5(\xi)$ are discontinuous, piecewise linear functions of their arguments, so that their integrals from $\xi = -1$ to $\xi = X_3$ are accordingly *continuous*, piecewise quadratic functions of X_3. Hence, for example, τ_{13} is a continuous function of position, with its value in the rth layer given by

$$\begin{aligned}\tau_{13}(X_1, X_3) &= \tau_{13}(X_1, Z_r) + \varepsilon^2 X_1 \int_{Z_r}^{X_3} N_4(\xi)\,\mathrm{d}\xi \\ &= \tau_{13}(X_1, Z_r) + \varepsilon^2 X_1 (X_3 - Z_r)\{L_1 Q_{11}^{(r)} + L_2 Q_{16}^{(r)} - \tfrac{1}{2}[X_3 + Z_r] Q_{11}^{(r)}\}\,p/E \end{aligned} \tag{49}$$

with the boundary value $\tau_{13}(X_1, 0) = 0$. Similarly, it can also be deduced from eqn (43) that the normal stress τ_{33} is a continuous, piecewise *cubic* function of X_3, with its value in the rth layer given by

$$\begin{aligned}\tau_{33}(X_3) &= \tau_{33}(Z_r) - \varepsilon^3 \frac{p}{E}(X_3 - Z_r) I(Z_r) \\ &\quad - \varepsilon^3 \frac{p}{E}(X_3 - Z_r)(X_3 + 2Z_r)\{L_1 Q_{11}^{(r)} + L_2 Q_{16}^{(r)} - \tfrac{1}{6}(X_3 - Z_r) Q_{11}^{(r)}\},\end{aligned} \tag{50}$$

where $I(Z_r)$ is defined by

$$I(Z_r) = \int_{-1}^{Z_r} \{L_1 Q_{11}(\xi) + L_2 Q_{16}(\xi) - \xi Q_{11}(\xi)\}\,\mathrm{d}\xi, \tag{51}$$

and is conveniently evaluated through the recursion relation

$$I(Z_{r+1}) = I(Z_r) + H_r\{(L_1 Q_{11}^{(r)} + L_2 Q_{16}^{(r)}) - \tfrac{1}{2}(Z_{r+1} + Z_r) Q_{11}^{(r)}\}, \tag{52}$$

with $I(Z_1) = I(-1) = 0$. In fact, the stresses themselves are obtained most conveniently by first determining their interlaminar values through recursion, and then computing their values within any particular layer by using the equations above. For example, the important interlaminar values of the shear stress τ_{13} satisfy

$$\tau_{13}(Z_{r+1}) = \tau_{13}(Z_r) + \varepsilon^2 \frac{p}{E} H_r X_1 \{L_1 Q_{11}^{(r)} + L_2 Q_{16}^{(r)} - \tfrac{1}{2}(Z_r + Z_{r+1}) Q_{11}^{(r)}\}. \tag{53}$$

Similar considerations show that U_1 and U_2 are continuous, piecewise *cubic* functions of X_3, whilst U_3 is piecewise *quartic* in X_3. Previous theories, including the "higher-order" laminate theories described by Christensen (1979), have *assumed* polynomial expressions for the displacements: however, it appears that none have assumed a piecewise cubic variation for U_1 and U_2 nor a piecewise quartic to approximate U_3. It is also important to note that the present method *automatically* ensures continuity of displacement and traction at every interface.

Finally, it should be noted that this solution is not only interesting in that it adds an exact analytical solution to the laminate literature, but also as a valuable test-case with which to investigate the usefulness of any software package for numerical solution of laminate problems.

7. EXAMPLE—SYMMETRIC CROSS-PLY LAMINATED STRIP

As an example, we consider a symmetric laminate in which each layer is of orthotropic material with the orthotropic axes parallel to the coordinate axes. Then in each lamina

$$c_{16} = c_{26} = c_{36} = c_{45} = 0, \quad Q_{16} = Q_{26} = Q_{36} = Q_{45} = 0,$$

and, for a symmetric laminate

$$\tilde{Q}_{ij} = 0, \quad L_1 = L_2 = 0.$$

Hence, from eqn (44)

$$N_3(\xi) = -\xi Q_{13}(\xi)p/E, \quad N_4(\xi) = -\xi Q_{11}(\xi)p/E, \quad N_5 = 0, \quad N_2 = 0,$$

and $N_1(\xi, \eta)$ and $N(\zeta, \eta, \zeta)$ are given by eqn (44). It follows from eqns (42) and (43) that

$$U_2 = 0, \quad \tau_{23} = 0.$$

Of the remaining stress and displacement variables, those of most interest are τ_{13} and τ_{33}, because the elementary classical solution gives no information about these quantities. They are also the stress components that have most influence on failure by delamination. For U_1, U_3, τ_{11}, τ_{22} and τ_{12} the leading order (in ε) terms in eqns (42) and (45) are independent of the laminate geometry, and usually the remaining terms give relatively small corrections to the leading terms. We therefore restrict consideration to τ_{13} and τ_{33}.

For the case under consideration the recurrence relation in eqn (53) for the interlaminar values of τ_{13} reduces to

$$\tau_{13}(Z_{r+1}) = \tau_{13}(Z_r) - \frac{\varepsilon^2 p}{2E} X_1 H_r (Z_r + Z_{r+1}) Q_{11}^{(r)}, \tag{54}$$

and, from eqn (50), the corresponding relation for interlaminar values of τ_{33} is

$$\tau_{33}(Z_{r+1}) = \tau_{33}(Z_r) - \varepsilon^3 \frac{p}{E} H_r I(Z_r) + \tfrac{1}{6}\varepsilon^3 \frac{p}{E} H_r^2 (Z_{r-1} + 2Z_r) Q_{11}^{(r)}, \tag{55}$$

with

$$I(Z_{r+1}) = I(Z_r) - \tfrac{1}{2} H_r (Z_{r+1} + Z_r) Q_{11}^{(r)}. \tag{56}$$

For definiteness we consider a laminate with N layers of equal thickness with alternating layers of two different orthotropic materials, such that

$$Q_{11}^{(r)} = Q_0 (r\,\text{odd}), \quad Q_{11}^{(r)} = Q_e (r\,\text{even}).$$

Since the laminate is symmetric, N is an odd integer. We now have

$$H_r = \frac{2}{N}, \quad Z_r = -1 + \frac{2(r-1)}{N},$$

and

Table 1

N	r	$\frac{E\tau_{13}(Z_r)}{\varepsilon^2 X_1 p}$	$-\frac{E\tau_{33}(Z_r)}{\varepsilon^3 p}$
3	1	0	0
	2	$0.444\,Q_0$	$0.173\,Q_0$
	3	$0.444\,Q_0$	$0.469\,Q_0+0.025\,Q_e$
	4	0	$0.642\,Q_0+0.025\,Q_e$
5	1	0	0
	2	$0.32\,Q_0$	$0.069\,Q_0$
	3	$0.32\,Q_0+0.16\,Q_e$	$0.197\,Q_0+0.037\,Q_e$
	4	$0.32\,Q_0+0.16\,Q_e$	$0.331\,Q_0+0.101\,Q_e$
	5	$0.32\,Q_0$	$0.459\,Q_0+0.139\,Q_e$
	6	0	$0.528\,Q_0+0.139\,Q_e$
7	1	0	0
	2	$0.245\,Q_0$	$0.037\,Q_0$
	3	$0.245\,Q_0+0.163\,Q_e$	$0.107\,Q_0+0.025\,Q_e$
	4	$0.327\,Q_0+0.163\,Q_e$	$0.190\,Q_0+0.072\,Q_e$
	5	$0.327\,Q_0+0.163\,Q_e$	$0.284\,Q_0+0.121\,Q_e$
	6	$0.245\,Q_0+0.163\,Q_e$	$0.367\,Q_0+0.167\,Q_e$
	7	$0.245\,Q_0$	$0.437\,Q_0+0.192\,Q_e$
	8	0	$0.474\,Q_0+0.192\,Q_e$
9	1	0	0
	2	$0.198\,Q_0$	$0.023\,Q_0$
	3	$0.198\,Q_0+0.148\,Q_e$	$0.067\,Q_0+0.017\,Q_e$
	4	$0.296\,Q_0+0.148\,Q_e$	$0.123\,Q_0+0.050\,Q_e$
	5	$0.296\,Q_0+0.198\,Q_e$	$0.188\,Q_0+0.090\,Q_e$
	6	$0.296\,Q_0+0.198\,Q_e$	$0.225\,Q_0+0.134\,Q_e$
	7	$0.296\,Q_0+0.148\,Q_e$	$0.321\,Q_0+0.173\,Q_e$
	8	$0.198\,Q_0+0.148\,Q_e$	$0.377\,Q_0+0.206\,Q_e$
	9	$0.198\,Q_0$	$0.421\,Q_0+0.223\,Q_e$
	10	0	$0.444\,Q_0+0.223\,Q_e$

$$\tau_{13}(Z_{r+1}) = \tau_{13}(Z_r) - 2\varepsilon^2 \frac{p}{EN^2}(N-2r+1)Q_{11}^{(r)}, \tag{57}$$

$$\tau_{33}(Z_{r+1}) = \tau_{33}(Z_r) - \frac{2\varepsilon^3 p}{EN} I(Z_r) - \frac{2}{3}\frac{\varepsilon^3 p}{EN^3}(3N-6r+4)Q_{11}^{(r)}, \tag{58}$$

$$I(Z_{r+1}) = I(Z_r) + \frac{2}{N^2}(N-2r+1)Q_{11}^{(r)}. \tag{59}$$

In Table 1 we give interlaminar values of τ_{13} and τ_{33} for $N = 3, 5, 7$ and 9, using the above formulae. Interlaminar values of the other stress and displacement components can be calculated in a similar way. Values of the stress and displacement components within the laminae are most easily found by interpolation, as outlined above.

Table 2. Values of τ_{13} at the midplane $X_3 = 0$

N	$\frac{\tau_{13}(0)E}{\varepsilon^2 X_1 p}$
1	$0.500\,Q_0$
3	$0.444\,Q_0+0.056\,Q_e$
5	$0.340\,Q_0+0.160\,Q_e$
7	$0.327\,Q_0+0.173\,Q_e$
9	$0.302\,Q_0+0.198\,Q_e$
11	$0.298\,Q_0+0.202\,Q_e$
13	$0.288\,Q_0+0.212\,Q_e$
15	$0.284\,Q_0+0.216\,Q_e$
∞	$0.250\,Q_0+0.250\,Q_e$

For the configuration described, the maximum value of τ_{13} always occurs at the mid-plane $X_3 = 0$. Values of τ_{13} at the mid-plane for various values of N are shown in Table 2.

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APPENDIX

Weighted integrals of the material moduli

For the leading order solution the only weighted integrals of the material moduli are $\bar{Q}_{ij}$, $\tilde{Q}_{ij}$ and $\hat{Q}_{ij}$, with

$$\{\bar{Q}_{ij}, \tilde{Q}_{ij}, \hat{Q}_{ij}\} = \int_{-1}^{1} \{1, \xi, \xi^2\}\, Q_{ij}(\xi)\, \mathrm{d}\xi \tag{A1}$$

as defined in eqns (22) and (23).

The next terms in the solution also require $\bar{Q}_{ij}$, $\tilde{Q}_{ij}$ and $\hat{Q}_{ij}$, but in addition they involve $\bar{R}_{\alpha\beta}$, $\tilde{R}_{\alpha\beta}$, $\bar{S}_{\alpha\beta}$, $\tilde{S}_{\alpha\beta}$, $\check{S}_{\alpha\beta}$ and $\hat{S}_{\alpha\beta}$ which are defined by weighted double and triple integrals:

$$\{\bar{R}_{\alpha\beta}, \tilde{R}_{\alpha\beta}\} = \int_{-1}^{1}\int_{-1}^{\xi} \{1, \xi\}\, R_{\alpha\beta}(\xi, \eta)\, \mathrm{d}\eta\, \mathrm{d}\xi$$

$$\{\bar{S}_{\alpha\beta}, \tilde{S}_{\alpha\beta}, \check{S}_{\alpha\beta}, \hat{S}_{\alpha\beta}\} = \int_{-1}^{1}\int_{-1}^{\xi}\int_{-1}^{\eta} \{1, \xi, \zeta, \xi\zeta\}\, S_{\alpha\beta}(\xi, \eta, \zeta)\, \mathrm{d}\zeta\, \mathrm{d}\eta\, \mathrm{d}\xi. \tag{A2}$$

Here $R_{\alpha\beta}$ and $S_{\alpha\beta}$ are the following combinations of the Q_{ij}:

$$\begin{aligned}
R_{11}(\xi, \eta) &= Q_{11}(\xi)Q_{44}(\eta) - Q_{16}(\xi)Q_{45}(\eta) - Q_{13}(\xi) \\
R_{12}(\xi, \eta) &= -Q_{11}(\xi)Q_{45}(\eta) + Q_{16}(\xi)Q_{55}(\eta) \\
R_{21}(\xi, \eta) &= Q_{16}(\xi)Q_{44}(\eta) - Q_{66}(\xi)Q_{45}(\eta) - Q_{36}(\xi) \\
R_{22}(\xi, \eta) &= -Q_{16}(\xi)Q_{45}(\eta) + Q_{66}(\xi)Q_{55}(\eta)
\end{aligned} \tag{A3}$$

and

$$\begin{aligned}
S_{11}(\xi, \eta, \zeta) &= R_{11}(\xi, \eta)Q_{11}(\zeta) + R_{12}(\xi, \eta)Q_{16}(\zeta) - Q_{11}(\xi)Q_{13}(\zeta) \\
S_{12}(\xi, \eta, \zeta) &= R_{11}(\xi, \eta)Q_{16}(\zeta) + R_{12}(\xi, \eta)Q_{66}(\zeta) - Q_{11}(\xi)Q_{36}(\zeta) \\
S_{21}(\xi, \eta, \zeta) &= R_{21}(\xi, \eta)Q_{11}(\zeta) + R_{22}(\xi, \eta)Q_{16}(\zeta) - Q_{16}(\xi)Q_{13}(\zeta) \\
S_{22}(\xi, \eta, \zeta) &= R_{21}(\xi, \eta)Q_{16}(\zeta) + R_{22}(\xi, \eta)Q_{66}(\zeta) - Q_{16}(\xi)Q_{36}(\zeta).
\end{aligned} \tag{A4}$$